Recall :

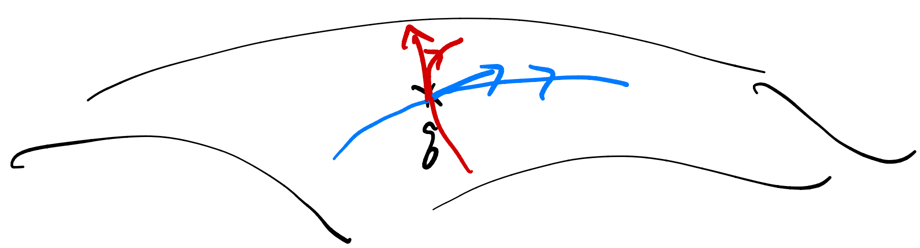
Given a regular surface $S \subseteq \mathbb{R}^3$, $p \in S$,
the tangent plane of S at p, denoted by T_pS
is given by $T_pS = \{ \alpha'(0) \in \mathbb{R}^3 \mid \alpha: (-\varepsilon, \varepsilon) \to S$ is diff., $\alpha(0) = p \}$

prop: $= dX_q(\mathbb{R}^2)$ if $X : U \to S$ is a parametrization where $X(q) = p$.

* Suppose $X(u,v) = (x(u,v), y(u,v), z(u,v))$

then $T_pS = \text{span}\{X_u(q), X_v(q)\}$

where $\begin{cases} X_u = (x_u, y_u, z_u) \\ X_v = (x_v, y_v, z_v) \end{cases}$ ← linearly indep. ⇒ dim = 2.



$\alpha(u) = (x(u, v_0), y(u, v_0), z(u, v_0))$, $X_u = \alpha'(u_0)$

$\beta(v) = (x(u_0, v), y(u_0, v), z(u_0, v))$, $X_v = \beta'(v_0)$

where $X(u_0, v_0) = p \in S$.

Change of coordinates: given $p \in S$,

let $X : U \to S$, $Y : V \to S$ be two parametrization of S at p.

i.e. $\begin{cases} X(u,v) = (x(u,v),\ y(u,v),\ z(u,v)) \\ Y(\eta,\xi) = (x(\eta,\xi),\ y(\eta,\xi),\ z(\eta,\xi)). \end{cases}$

$\boxed{\text{WLOG, assume } \underset{\text{(translation)}}{(0,0)} \xmapsto{X,\,Y} p}$

meaning ??

then $T_pS = \operatorname{span}\{X_u, X_v\}\big|_{(0,0)} = \operatorname{span}\{Y_\eta, Y_\xi\}\big|_{(0,0)}$

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given $w \in T_pS$, $w = a X_u + b X_v$

or $= c Y_\eta + d Y_\xi$.

??

$$X_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$= \left(\frac{\partial x}{\partial \eta}\cdot\frac{\partial \eta}{\partial u} + \frac{\partial x}{\partial \xi}\cdot\frac{\partial \xi}{\partial u},\ \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial u} + \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial u},\ \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial u} + \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial u}\right)$$

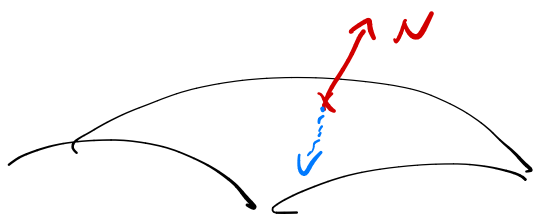
$$= \frac{\partial \eta}{\partial u}\cdot\left(\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}, \frac{\partial z}{\partial \eta}\right) + \frac{\partial \xi}{\partial u}\cdot\left(\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}, \frac{\partial z}{\partial \xi}\right) = \frac{\partial \eta}{\partial u}\cdot\frac{\partial Y}{\partial \eta} + \frac{\partial \xi}{\partial u}\cdot\frac{\partial Y}{\partial \xi}.$$

$$\Rightarrow \underbrace{\boxed{\begin{bmatrix} \frac{\partial \eta}{\partial u} & \frac{\partial \xi}{\partial u} \\ \frac{\partial \eta}{\partial v} & \frac{\partial \xi}{\partial v} \end{bmatrix}}}_{= d(Y\circ X^{-1})|_{(0,0)} = \text{invertible } (\because \text{diff.})} \begin{bmatrix} Y_\eta \\ Y_\xi \end{bmatrix} = \begin{bmatrix} X_u \\ X_v \end{bmatrix}.$$

Defn: let S be a regular surface and $p \in S$.

A non-zero vector N at p is called a normal vector of S at p if $N \perp v$, $\forall v \in T_pS$.

N is unit normal vector if $|N| = 1$.



Q: how to find it??

(A) If $X : \mathcal{U} \to S$ is a parametrization of S at p

then $T_pS = \text{span}\{X_u(q), X_v(q)\}$ when $X(q) = p$.

$\Rightarrow$ $X_u(q) \times X_v(q) =$ vector in $\mathbb{R}^3$ $\perp T_pS$

$\Rightarrow$ $N = \pm \left.\dfrac{(X_u \times X_v)}{\| X_u \times X_v \|}\right|_q$ is the unit normal to S at p.

(B) If $S = f^{-1}(c)$ for some differentiable function f and regular value $c \in \mathbb{R}$,

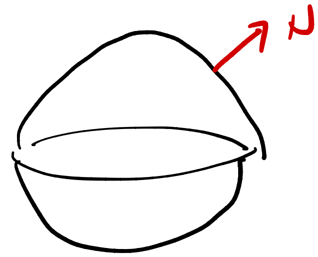
then $N = \pm \dfrac{\nabla f}{\|\nabla f\|}$.

Since if $\alpha : (-\varepsilon, \varepsilon) \to S = f^{-1}(c)$, $\alpha(0) = p$

then $f \circ \alpha(t) \equiv c$

$\Rightarrow \langle \nabla f, \alpha' \rangle = 0$ at $t=0$ #

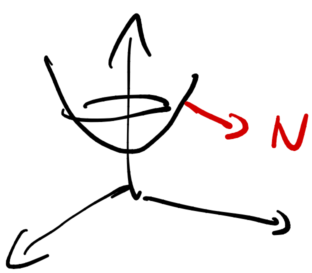
eg:

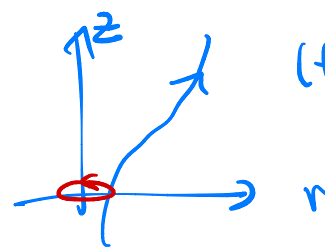
① 

$$S^2 = f^{-1}(1) \quad \text{where} \quad f = x^2+y^2+z^2$$

$$\Rightarrow \nabla f = (2x, 2y, 2z)$$

$$\Rightarrow N = \pm(x, y, z)$$

② 

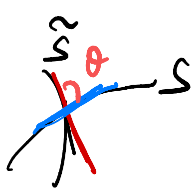


$(f(u), g(u))$, $f>0$.

$$X(u,v) = \left(f(u)\cos v,\ f(u)\sin v,\ g(u)\right)$$

$$\begin{cases} X_u = (f'\cos v,\ f'\sin v,\ g'(u)) \\ X_v = (-f\sin v,\ f\cos v,\ 0) \end{cases}$$

$$\Rightarrow \begin{cases} X_u \times X_v = (-fg'\cos v,\ -fg'\sin v,\ ff') \\ \|X_u \times X_v\| = \sqrt{(fg')^2+(ff')^2} = f\sqrt{(g')^2+(f')^2} \end{cases}$$



θ = angle between N, $\tilde{N}$.

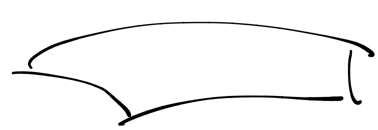
s.t.

$$N = \frac{\pm(-g'\cos v,\ -g'\sin v,\ f')}{\sqrt{(g')^2+(f')^2}}$$

#

1st fundamental form (or metric) g.

Motivation: given a regular surface S.



given a parametrized curve $\gamma : I \to S \subseteq \mathbb{R}^3$ where $\gamma(0) = p \in S$

$\gamma'(0) \in T_pS$ and $\gamma'(t) =$ velocity

$\Rightarrow$ speed of $\gamma' = \|\gamma'(0)\|$.

Q: To compute $\|\gamma'(0)\|$, how??

$\because \gamma(t) \in S \quad \forall t \in I$

$\Rightarrow \gamma(t) = X \circ \alpha(t)$ where $X =$ parametrization of S at p.

$\Rightarrow \gamma'(0) = dX_q(\alpha'(0)) = dX_q(a e_1 + b e_2)$

$= a X_u + b X_v$.

$$\therefore \|\gamma'(0)\|^2 = \langle a X_u + b X_v , a X_u + b X_v \rangle$$

$$= a^2 \langle X_u, X_u \rangle + 2ab \langle X_u, X_v \rangle + b^2 \langle X_v, X_v \rangle.$$

$$= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

View as a bilinear form on T_pS.

called this the 1st f.f. g.

(Equivalently)

Defn: Given a regular surface $S \subseteq \mathbb{R}^3$, $p \in S$.

The first fundamental form g of S at p is given by $g_p(v, w) \triangleq \langle v, w \rangle_{\mathbb{R}^3}$ for all $v, w \in T_pS$.

This is an inner product on the vector space T_pS.

$\ast$ So given a parametrization $X : \mathcal{U} \to S$.

wrt basis of $T_pS = \text{span}\{X_u, X_v\}$.

$$[g] = \begin{bmatrix} g(X_u, X_u) & g(X_u, X_v) \\ g(X_u, X_v) & g(X_v, X_v) \end{bmatrix} \triangleq \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

$\star$ $g_p(v,v) = \langle v, v\rangle_{\mathbb{R}^3} \geq 0$, and "$=$" $\Leftrightarrow$ $v \equiv 0$.

$\because$ X is differentiable.

$\therefore$ E, F, G are differentiable on $\mathcal{U}$.

$\Rightarrow$ the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is differentiable locally.

Why care?? $\rightarrow$ length (speed on curve on S)

$\rightarrow$ Area

$\rightarrow$ angle

$\star\star$: $\{E, F, G\}$ contains all local geometric information (later)

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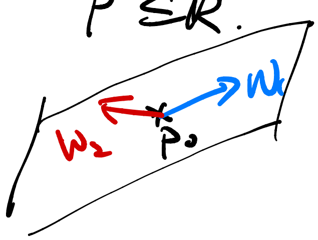
eg: $P \subseteq \mathbb{R}^3$.



$p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$

parametrization of P:

$$X(u,v) = p_0 + u W_1 + v W_2$$

where $W_1, W_2 \in T_pP$ which are linearly independent.

Gram - Schmidt process

$\Rightarrow \exists\ \tilde{w}_1, \tilde{w}_2 \in \operatorname{span}\{w_1, w_2\} = T_pP$

s.t. $\{\tilde{w}_1, \tilde{w}_2\}$ are orthonormal.

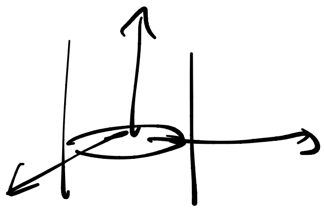
Reparametrise P by

$$\tilde{X}(u,v) = p_0 + u\,\tilde{w}_1 + v\,\tilde{w}_2 .$$

$$\Rightarrow \begin{cases} \tilde{X}_u = \tilde{w}_1 \\ \tilde{X}_v = \tilde{w}_2 . \end{cases} \quad \text{s.t. } [g_p] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{I_2} \quad \#$$

compare

eg 2



cylinder : $x^2+y^2=1$.

$$X(u,v) = (\cos u, \sin u, v)$$

$$\Rightarrow \begin{cases} X_u = (-\sin u, \cos u, 0) \\ X_v = (0, 0, 1) \end{cases} \Rightarrow \{X_u, X_v\} \text{ are o.n.}$$

SAME.

$$\Rightarrow [g_p] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{I_2}$$

* globally different
* locally same even on the level of Isometry!!

eg3: helicoid (see picture from wiki)

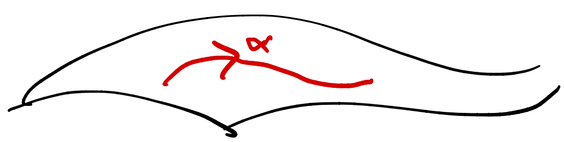
$a>0$ is fixed

$X(u,v) = (v\cos u, v\sin u, au)$, $u \in (0,2\pi)$, $v \in \mathbb{R}$

$$\begin{cases} X_u = (-v\sin u, \ v\cos u, \ a) \\ X_v = (\cos u, \ \sin u, \ 0) \end{cases}$$

$$\Rightarrow \begin{cases} \langle X_u, X_u \rangle = v^2 + a^2 \\ \langle X_u, X_v \rangle = 0 \\ \langle X_v, X_v \rangle = 1 \end{cases} \Rightarrow [g_p] = \begin{bmatrix} v^2+a^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose $\alpha: I \to S \subseteq \mathbb{R}^3$ is a parametrized curve with $\alpha' \neq 0$



$\alpha'(t)$ = velocity. hence

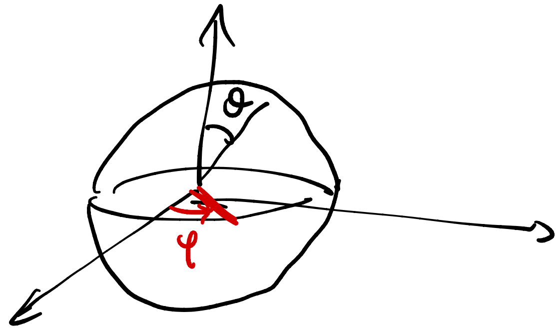
$$s(t) = \int_0^t \|\alpha'(\tau)\| \, d\tau = \int_0^t \sqrt{g(\alpha', \alpha')} \, d\tau$$

(= arc-length from $\tau = 0$ to $\tau = t$)

$$= \int_0^t \sqrt{E(u')^2 + 2F u'v' + G(v')^2} \, d\tau$$

where $\alpha(t) = X(u(t), v(t))$ for some parametrization $X: \mathcal{U} \to S \subseteq \mathbb{R}^3$.

eg:

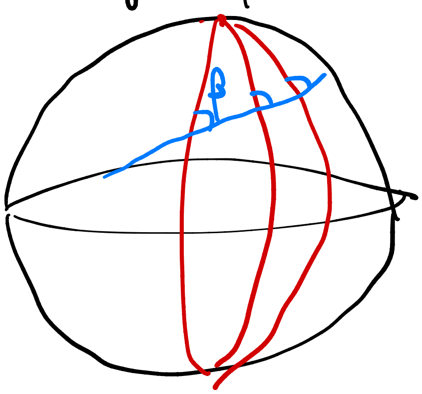


$$X(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

then $\begin{cases} X_\theta = (\cos\theta\cos\varphi, \ \cos\theta\sin\varphi, \ -\sin\theta) \\ X_\varphi = (-\sin\theta\sin\varphi, \ \sin\theta\cos\varphi, \ 0) \end{cases}$

$$\Rightarrow \quad E = 1, \quad F = 0, \quad G = \sin^2\theta$$

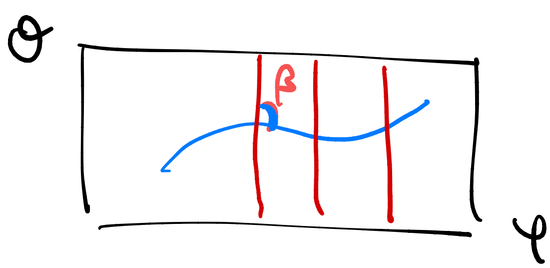
Now, we are looking for the curve on S^2 which makes constant angle β with $\varphi \equiv$ const. (called Rhumb line)



Suppose the curve $\alpha(t)$ = image of $(\theta(t), \varphi(t))$ via X.

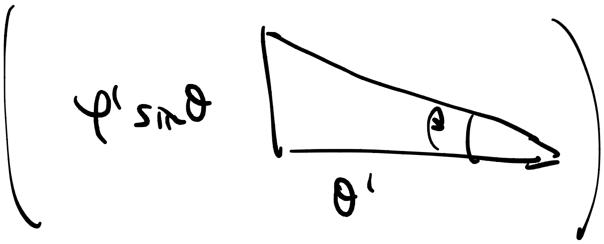
$$\alpha' = X_\theta \cdot \theta' + X_\varphi \varphi'$$

assumption $\Rightarrow$ at each $t_0 \in I$.



$$\frac{\langle \alpha', X_\theta \rangle}{\|\alpha'\| \|X_\theta\|} = \cos\beta$$

$$\Rightarrow \cos\beta = \frac{\theta'}{\sqrt{(\theta')^2 + (\varphi')^2\sin^2\theta}} \Rightarrow \cos^2\beta = \frac{(\theta')^2}{(\theta')^2 + (\varphi')^2\sin^2\theta}$$



$$\frac{(\varphi' \sin\theta)^2}{(\theta')^2} = \tan^2\beta$$

orientation of curve.

$$\Rightarrow \quad \frac{\theta'}{\sin\theta} = \pm \frac{\varphi'}{\tan\beta}$$

$$\Rightarrow \quad \log\tan(\theta/2) = \pm(\varphi \pm c)\cot\beta \qquad \#.$$

Area of a region:

given $X : \mathcal{U} \to S$, a parametrization of a regular surface S.

Let R = open region in S s.t. $X(V) = R$

then

$$\text{Area}(R) \triangleq \int_R dA \leftarrow \text{area element on } S$$

$$= \int_V \|X_u \times X_v\| \, du\, dv$$

$$= \int_V \underbrace{\sqrt{EG - F^2}}_{\text{morally the size of Jacobian.}} \, du\, dv$$